Active and Passive Third-Order Filters

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Simple and general design equations are given for active and passive third-order filters. Also a variety of optimal prototype-designs are presented.

Introduction: Simple design equations for active and passive third-order filters are given. We consider third order lowpass filters which have a transfer function of the form

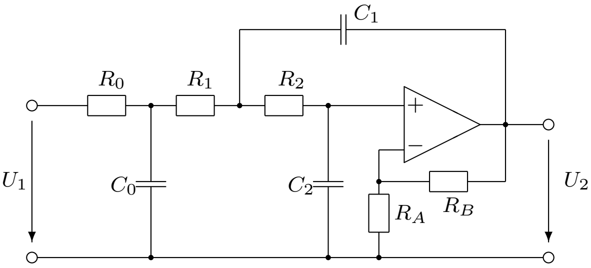
$$H(s) = \frac{K}{a_3\,s^3 + a_2\,s^2 + a_1\,s + 1}\ .$$

The coefficients a_1, a_2, a_3 are positive real numbers and K is a non-zero real number. The transfer function $H(s)$ is called allpole transfer function because in the numerator we have a constant. For the active filters we consider the designs given in [3] and [1] and for the passive filters the classical singly and doubly terminated LC-ladder designs. We will express the relevant parameter values as function of the values a_3, a_2, and a_1 and as function of the values b_3, b_2, and b_1 defined from the magnitude

$$\sqrt{H(\imath\omega)H(-\imath\omega)} = \frac{K}{\sqrt{1 + b_3\,\omega^6 + b_2\,\omega^4 + b_1\,\omega^2}}\ .$$

Third-Order Active Allpole Lowpass Filters: An active circuit which realises the above transfer function is displayed in figure 1. It consists of a first order RC-lowpass followed by a second order Sallen-Key block. For simplicity we use normalised values. Standard circuit analysis leads to

Fig. 1. Third Order Allpole Lowpass



$$a_3 = R_0R_1R_2C_0C_1C_2$$

$$a_2 = R_0R_1C_0C_1(1-K) + R_0R_2C_0C_2 + R_0R_1C_0C_2 + R_0R_2C_1C_2 + R_1R_2C_1C_2$$

$$a_1 = R_0C_0 + R_0C_1(1-K) + R_0C_2 + R_1C_1(1-K) + R_2C_2 + R_1C_2$$

where $K = \frac{R_A+R_B}{R_A}$, i.e. in this circuit K is a real number greater or equal to unity. In practice one usually prefers simple designs in which as many components as possible have equal values. Therefore Huelsman's approach [3] was to set all capacitances to unity. It turned out that for the simplest case $K=1$, i.e. the case of a voltage follower, this does do not lead to implementations for many practical cases. Huelsman therefore used $K=2$ which lead to realisable designs not only for Butterworth [3] but also for other common filter characteristics [4],pp.286-287. Huelsman found his solutions by numerical techniques. For $C_0 = C_1 = C_2 = 1$ and $K=2$ we get $(a_3, a_2, a_1) = (R_0R_1R_2, 2R_0R_2 + R_1R_2, R_0 + R_2)$ and a straight-forward calculation then gives the value of R_0 as real root of the third degree polynomial $R_0^3 - a_1R_0^2 + \frac{a_2}{2}R_0 - \frac{a_3}{2} = 0$ and the values of $R_2 = a_1 - R_0$ and $R_1 = \frac{a_3}{R_0(a_1-R_0)}$ follow. Application of a well-known formula for the solution of cubic equations then for $a_2 < \frac{2}{3}a_1^2$ leads to

$$R_0 = \frac{2}{3}\sqrt{a_1^2 - \frac{3}{2}a_2}\cdot\cosh\left(\frac{1}{3}\,\mathrm{acosh}\left(\frac{a_1^3 - \frac{9}{4}a_1a_2 + \frac{27}{4}a_3}{(a_1^2 - \frac{3}{2}a_2)^{\frac{3}{2}}}\right)\right) + \frac{a_1}{3}.$$

For instance for a Butterworth filter we have $(a_3, a_2, a_1) = (1, 2, 2)$ which leads to $R_0 = \frac{2}{3}\cosh(\frac{1}{3}\cosh^{-1}(\frac{23}{4})) + \frac{2}{3} \approx 1.5652$ and the values $R_2 = a_1 - R_0 \approx 0.4348$ and $R_1 = \frac{a_3}{R_0R_2} \approx 1.4694$ follow in agreement with the numerical data in [3].

As another example consider a Tschebyscheff filter with passband ripple ϵ. Transfer function and poles (see e.g. [2], p.27 and p.9) are given by $H(s) = \frac{K}{4\epsilon(s-p_1)(s-p_2)(s-p_3)}$ and $p_{1/3} = -\frac{1}{2}\sinh(\frac{1}{3}\mathrm{asinh}(\frac{1}{\epsilon})) \pm \imath\frac{\sqrt{3}}{2}\cosh(\frac{1}{3}\mathrm{asinh}(\frac{1}{\epsilon}))$, $p_2 = -\sinh(\frac{1}{3}\sinh^{-1}(\frac{1}{\epsilon}))$. This gives $(a_3, a_2, a_1) = (4\epsilon, 8\epsilon\sigma, \epsilon(8\sigma^2+3))$, where $\sigma = \sinh(\frac{1}{3}\mathrm{asinh}(\frac{1}{\epsilon}))$, $K=2$ and the equation for R_0 can be used.

Often a desired magnitude function is given, i.e. we are given the triple (b_3, b_2, b_1). Setting $s^2 = -\omega^2$ from magnitude and transferfunction we get the factorisation $(a_3s^3 + a_2s^2 + a_1s + 1)\cdot(-a_3s^3 + a_2s^2 - a_1s + 1) = -b_3s^6 + b_2s^4 - b_1s^2 + 1$. Comparison of terms yields

$$a_3^2 = b_3, \quad -2a_3\,a_1 + a_2^2 = b_2, \quad a_1^2 - 2a_2 = b_1\ .$$

From these equations we find that a_2 is root of the fourth degree equation

$$x^4 - 2b_2\cdot x^2 - 8b_3\cdot x + (b_2^2 - 4b_1b_3) = 0\ . \tag{1}$$

This means that we can change from (a_3, a_2, a_1) to (b_3, b_2, b_1) and back (namely $a_3 = \sqrt{b_3}$, a_2 is root of (1), and $a_1 = \sqrt{b_1 + 2a_2}$). A common approach in filter design which encompasses Butterworth and Tschebyscheff filters is to set $b_3\omega^6 + b_2\omega^4 + b_1\,\omega^2 = \epsilon^2\cdot(u\omega^3 - v\omega)^2$. With this approach we get $b_2^2 = 4b_1b_3$ and the quartic for a_2 reduces to a cubic which can be solved as

$$a_2 = 2\sqrt{\frac{-2b_2}{3}}\,\sinh\left(\frac{1}{3}\,\mathrm{asinh}\left(\frac{6b_3}{-b_2\sqrt{-b_2}}\sqrt{\frac{3}{2}}\right)\right)$$

which using u and v yields $a_2 = \sqrt{\frac{uv}{3}}\,4\epsilon\,\sinh\left(\frac{1}{3}\,\mathrm{asinh}\left(\frac{3}{2v\epsilon}\sqrt{\frac{3u}{v}}\right)\right)$. As a check, for the third Tschebyscheff polynomial $4w^3 - 3\omega$ we can recover the value $a_2 = 8\epsilon\sinh(\frac{1}{3}\mathrm{asinh}(\frac{1}{\epsilon}))$ given above.

It is now very simple to design various optimal filters.

To obtain a least mean square filter ([5], p.381) we set $\epsilon^2\cdot(u\,\omega^3 - v\,\omega)^2 = \epsilon^2\cdot(\frac{7}{4}\omega^3 - \frac{3}{4}\omega)^2$, i.e. $u = \frac{7}{4}$ and $v = \frac{3}{4}$. In table 1 below the values for the resulting resistors are given for $\epsilon = 1$ in the row LMS.

The normalised polynomials $P_n(x)$ of degree n which minimise the integral $\int_{-1}^{1}|P_n(x)|dx$ are the Tschebyscheff polynomials of the second kind ([7], pp.70-71) which follow up to a multiplicative constant as the derivative of the (usual) Tschebyscheff polynomials of the first kind. For our case $n=3$ we therefore set $\epsilon^2\cdot(u\,\omega^3 - v\,\omega)^2 = \epsilon^2\cdot(2\omega^3 - \omega)^2$, i.e. $u=2$ and $v=1$. In table 1 below the values of the resulting resistors are given for $\epsilon=1$ in the row Tschebyscheff II.

A further sensible selection are Legendre polynomials. These polynomials minimise the average distance to zero (see [7], p.71). For degree $n=3$ this leads to $\epsilon^2\cdot(u\,\omega^3 - v\,\omega)^2 = \epsilon^2\cdot(\frac{5}{2}\omega^3 - \frac{3}{2}\omega)^2$, i.e. $u = \frac{5}{2}$ and $v = \frac{3}{2}$. Again in table 1 the values of the resulting resistors are given for $\epsilon = 1$. We call these filters class-LII filters to distinguish them from the class-L filters considered next.

In [6] the class-L filters were introduced which are also related to Legendre polynomials. Their amplitude response is monotonic and has the sharpest cut-off rate which an allpole filter can have. For degree $n=3$ we get $b_3\omega^6 + b_2\omega^4 + b_1\omega^2 = \epsilon^2(3\omega^6 - 3\omega^4 + \omega^2)$. For generality the ripple ϵ is used, Papoulis considered $\epsilon = 1$. This time the quartic for a_2 does not reduce to a cubic but by applying the solution formulas for quartics one sees that the associated cubic resolvent has a particularly simple form, i.e. the distinctive feature of the class-L filters finds its algebraic counterpart in a particular resolvent $(z+\epsilon^2)^3 = \epsilon^4(\epsilon^2+9)$ from which a_2 follows. For $\epsilon = 1$ we obtain

$$a_2 = \sqrt{10^{\frac{1}{3}} - 1} + \sqrt{2}\cdot\sqrt{\sqrt{10^{\frac{2}{3}} + 10^{\frac{1}{3}} + 1} - \frac{10^{\frac{1}{3}}}{2} - 1} = 2.270204\ldots$$

and the resistor values (displayed in table 1) follow.

The condition $a_2 < \frac{2}{3}a_1^2$ yields $a_2 > -2b_1$. Since by Hurwitz's criterion a_2 must be a positive number this condition is fulfilled for $b_1 \geq 0$, i.e. for all designs of the form $\epsilon^2(u\omega^3 - v\omega)^2$. The entries in table 1 are ordered according to their slope $\frac{d}{d\omega}(b_3\omega^6 + b_2\omega^4 + b_1\omega^2)|_{\omega=1} = 6b_3 + 4b_2 + 2b_1$.

Table 1: Lowpass designs, equal capacitor case, $\epsilon = 1$, $K = 2$

	R_0	R_1	R_2	slope
Butterworth	1.5652	1.4694	0.4348	6
Class-L	2.003	2.465	0.3508	8
LMS	1.9122	2.5054	0.3653	9
Tschebyscheff II	2.0540	2.8874	0.3372	10
Class-LII	2.3724	3.6992	0.2849	12
Tschebyscheff	3.5334	6.3886	0.1772	18

Geffe [1] treated the case $R_0 = R_1 = R_2 = 1$ with $K=1$ and gave a cubic for C_2. We use the cubic $C_0^3 - a_1C_0^2 + \frac{3}{2}a_2C_0 - 3a_3 = 0$ which can be derived and solved as the cubic for R_0. For $a_2 > \frac{2}{9}a_1^2$ we find

$$C_0 = \frac{2}{3}\sqrt{\frac{9}{2}a_2 - a_1^2} \cdot \sinh\left(\frac{1}{3}\,\mathrm{asinh}\left(\frac{a_1^3 - \frac{27}{4}a_1a_2 + \frac{81}{2}a_3}{(\frac{9}{2}a_2 - a_1^2)^{\frac{3}{2}}}\right)\right) + \frac{a_1}{3}\ ,$$

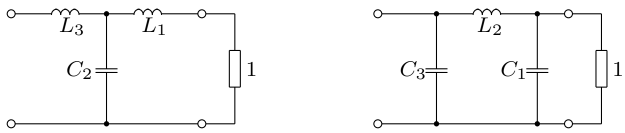
and $C_1 = 3a_3/(C_0(a_1 - C_0))$ as well as $C_2 = (a_1 - C_0)/3$ follow. In table 2 the same designs as in table 1 are given with exception of the Tschebyscheff case for $\epsilon = 1$ which is not realisable. This follows from the condition $a_2 > \frac{2}{9}a_1^2 \Rightarrow a_2 > \frac{2}{5}b_1$ which is violated for $\epsilon = 1$.

Table 2: Lowpass designs, equal resistor case, $\epsilon = 1$, $K = 1$

	C_0	C_1	C_2
Butterworth	1.3926	3.5468	0.20245
Class-L	1.9660	6.8158	0.12925
LMS	1.9041	7.3841	0.1245
Tschebyscheff II	2.0779	9.2160	0.1044
Class-LII	2.4387	14.076	0.072827

Third Order Passive Allpole Lowpass Filters: Using the transfer function and our knowledge of the values a_3, a_2, a_1, it is a straight-forward procedure to determine passive LC-realisations for the filters considered above. The simplest cases are the two singly terminated lossless ladder networks depicted in figure 2. For the singly terminated cases we find

Fig. 2. Singly Terminated Third Order Lowpass Filters



$(L_1, C_2, L_3) = (\frac{a_3}{a_2}, \frac{a_2}{a_1 - \frac{a_3}{a_2}}, a_1 - \frac{a_3}{a_2}) = (C_1, L_2, C_3)$, where $K = 1$.
For the same indexing the doubly (unity) terminated cases yield $L_3 = \frac{2a_3}{a_2 - \eta}$, $C_2 = a_1 - \sqrt{b_1}$, and $L_1 = \frac{a_2 - \eta}{a_1 - \sqrt{b_1}}$ where $\eta = \sqrt{b_2 + 2\sqrt{b_1}\sqrt{b_3}}$. For $\eta = 0$, i.e. $b_2 = -2\sqrt{b_1}\sqrt{b_3}$ we get $L_1 = L_3$ or as function of u and v introduced above $L_3 = L_1 = \sqrt{\frac{3u}{4v}}\,/\sinh\left(\frac{1}{3}\,\mathrm{arsinh}\left(\frac{3}{2v\epsilon}\sqrt{\frac{3u}{v}}\right)\right)$, and $C_2 = \frac{8}{3}\epsilon v \cdot \sinh^2\left(\frac{1}{3}\,\mathrm{arsinh}\left(\frac{3}{2v\epsilon}\sqrt{\frac{3u}{v}}\right)\right)$. Clearly, $K = \frac{1}{2}$.
For the second form we replace $L_3 \rightarrow C_3$, $C_2 \rightarrow L_2$, and $L_1 \rightarrow C_1$.

Conclusion: Very simple design equations for third order prototype filters have been given. Setting $v = u - 1$ allows the design of filters having ripple ϵ and slope $\epsilon^2(4u+2)$ for $1 \leq u \leq 4$. Also various optimal filter characteristics have been considered which can be extended for any degree n. From the lowpass prototypes highpass, bandpass and bandstop designs follow using standard techniques. For Butterworth and Tschebyscheff filters, which are special cases of Cauers filters, the pole locations are circles and ellipses. Recently [2] the pole locations of Cauer filters have been shown to be related in a simple way to Cartesian ovals. We are confident that for the remaining optimal designs considered here for $n = 3$, simple curves of the pole locations can be determined for any order.

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